UDC 621. 396. 677. 3:
621. 375

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RESEARCH DEPARTMENT

THE QUEEN'S AWARD TO INDUSTRY 1969

REPORT

Active aerial arrays: reciprocity

No. 1971/17

Research Department, Engineering Division
THE BRITISH BROADCASTING CORPORATION

RESEARCH DEPARTMENT

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Work covered by this report was undertaken by the BBC Research Department for the BBC and the ITA

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May 1971

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ACTIVE AERIAL ARRAYS : RECIPROCITY

Summary

It is shown that the usual Reciprocity Theorem does not hold in general for active aerial arrays. A new formulation of reciprocity is presented which can be applied to active aerials.

1. Introduction

The theorem of reciprocity is important in the theory and design of conventional aerial arrays. Considering a particular array, it is often easier to calculate its radiation pattern as a transmitting array from first principles than to calculate its directivity pattern as a receiving array. Reciprocity, however, shows that the two patterns are identical functions of direction and therefore the more difficult calculation may be avoided. It is tempting to hope that this principle might also be applied to the study of active receiving arrays.* It has previously been employed[1] to calculate patterns for linear arrays of identical active aerial elements but it is shown in this report that in the more general case it is not valid unless a certain constraint is met. Without the constraint a more complicated reciprocity statement can be applied which, nevertheless, permits pattern calculations for receiving arrays to be based on the transmitting case.

2. The theorem of reciprocity

The reciprocity theorem in its black box form is illustrated in Fig. 1. D is a black box containing linear elements in which circuit loops A and B have been exposed. (The series impedance elements Z_A and Z_B are not necessary for the statement of the theorem. They have been included for consistency with the proof given in Section 4).

* Aerial arrays with integrated active elements — e.g. transistor amplifiers.

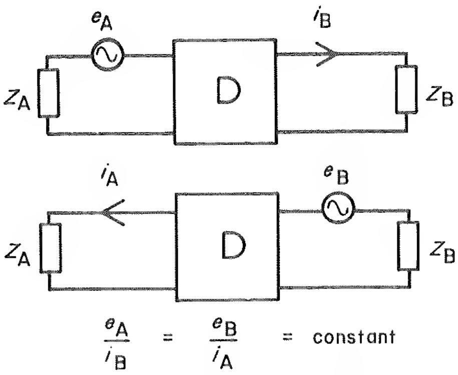

Fig. 1 - The theorem of reciprocity

A single frequency e.m.f. e_A is introduced into loop A and a resultant current i_B flows in loop B. Reciprocally an e.m.f. e_B (at the same frequency as e_A) is introduced into loop B and a resultant current i_A flows in loop A. Then,

$$e_A/i_B = e_B/i_A = \text{constant}$$

If D contains only linear lumped circuit elements the proof of the theorem is straightforward.[2] If D includes more general types of linear constituents such as transmission lines, aerials, propagation paths and scattering elements, the theorem still holds[3] and it is for this reason that it is of fundamental importance in aerial theory.

3. Amplifiers

The aerial systems under consideration are active in that amplifiers are integrated into the array. For example the array might comprise a number of dipole elements, each dipole having a small solid state amplifier placed directly across its terminals.[1] For convenience it is assumed that all amplifiers in the array act as ideal buffer stages although this requirement is not strictly essential. The case of more general amplifier types is discussed in the Appendix. Fig. 2 shows the two configurations which are specified for the amplifiers designated A to B and B to A respectively. The rth amplifier G_r has infinite input impedance, zero output impedance and a gain g_r defined as open-current output p.d./input p.d. In the A to B configuration, the arrangement in Fig. 2(*a*) acts like a buffer amplifier with input impedance Z_{Ar} and output impedance Z_{Br} and vice versa in the reverse B to A configuration. [Fig. 2(*b*)].

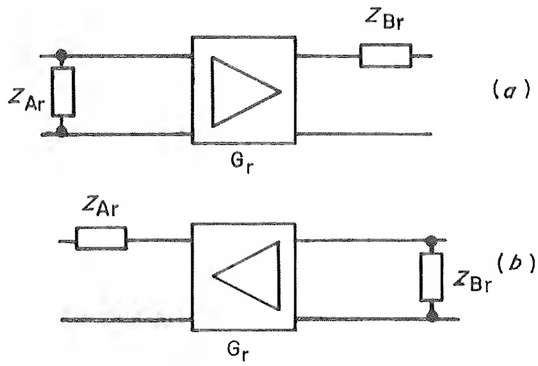

Fig. 2 - Buffer amplifier schematic
(*a*) Schematic buffer amplifier connected A to B
(*b*) Schematic buffer amplifier connected B to A

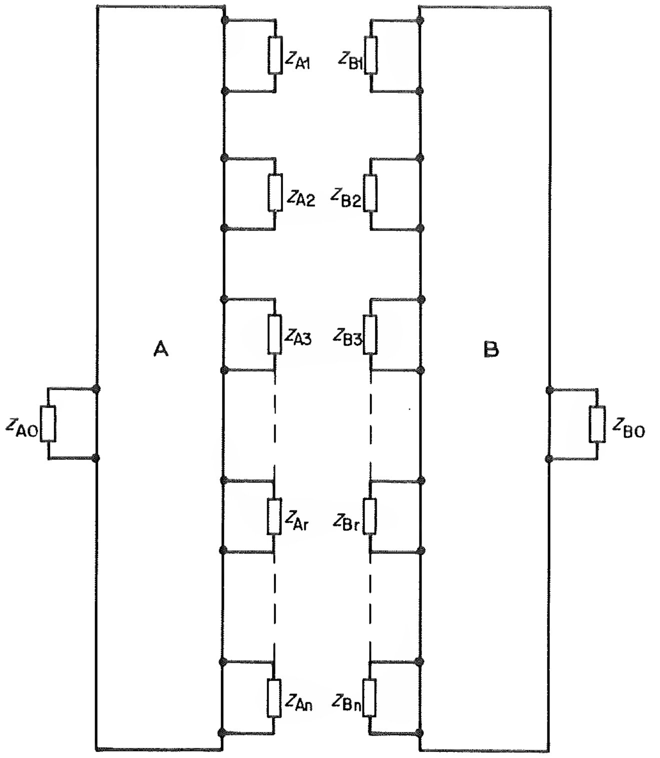

Fig. 3 - The two linear systems A and B

4. Interconnection of linear systems by buffer amplifiers

Fig. 3 shows two passive linear systems A and B in each of which (n + 1) circuit loops have been exposed. Each loop includes an impedance, numbered as shown. The systems A and B are of the general type mentioned in Section 2. Now let amplifiers be inserted between impedances Z_{A1} and Z_{B1}, Z_{A2} and Z_{B2} etc. so that Z_{A1} and Z_{B1} form the input and output impedances of the first amplifier connected in the A to B configuration (Fig. 2) and so on.* Let an e.m.f. e in series with Z_{A0} result in a current $A_1 e$ flowing in Z_{A1}, a current $A_2 e$ flowing in Z_{A2} etc. The amplifiers will consequently introduce effective e.m.f.s equal to $A_1 e Z_{A1} g_1$, $A_2 e Z_{A2} g_2$ etc. in series with Z_{B1}, Z_{B2} etc. respectively. Let an e.m.f. e in series with Z_{B1} cause a current $B_1 e$ to flow in Z_{B0} and similarly for the other loops. Applying the superposition principle therefore, the current I in Z_{B0} due to the e.m.f. e in series with Z_{A0} is

$$I = e\sum A_r B_r g_r Z_{Ar} \quad ** \qquad (1)$$

Exactly similar arguments apply in the reverse direction when all the amplifiers are connected in the B to A configuration when an e.m.f. e is introduced in series with Z_{B0} leading to a current I^1 in Z_{A0}

$$I^1 = e\sum B_r A_r g_r Z_{Br} \qquad (2)$$

* All quantities are complex in this report.

** All summations in the report run from r = 1 to r = n.

All the quantities A_r and B_r are identical in the two cases by reciprocity and the g_r are identical by hypothesis. Thus the currents I and I^1 are proportional if $Z_{Ar} = KZ_{Br}$ (for all r), where K is a constant.

5. General application of the reciprocity principle

It follows from the foregoing section that, if the input and output impedances of all the amplifiers in the array have a constant ratio, the transmitting and receiving patterns of the array are proportional. To calculate the transmitting pattern of a receiving array it is only necessary to reverse the amplifier directions as previously explained. However, it may be possible to use Equations (1) and (2) to derive the receiving pattern from the transmitting pattern when the amplifier impedances are not restricted in value. This requires a knowledge of the A_r and the B_r. Theoretically the two patterns may exhibit marked differences. An example is given in the following section of a particular case in which the A_r and the B_r can be calculated.

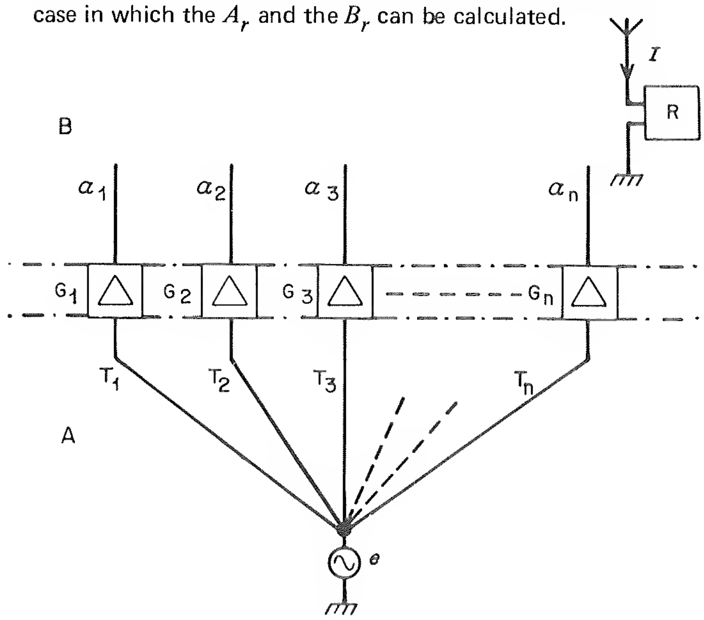

Fig. 4 - Active transmitting array, schematic

6. An example of the application of the reciprocity principle

Fig. 4 represents a simple active transmitting aerial comprising n radiating elements (α_1, α_2 ..... α_n) driven by amplifiers G_1 etc. The inputs to the amplifiers are connected to a common e.m.f. source e through transmission lines T_1 etc. The free-space far field is picked up by the receiver R which has a current I flowing into its input terminals. The bottom of the figure corresponds to the linear system A of Section 4, the top corresponds to system B. The A_r factors in this case can easily be found from the lengths and characteristics of the input transmission lines and the amplifier input impedances. It is assumed that each amplifier has a known gain and that the far field at the receiver and the resultant receiver aerial current can be calculated in the usual way from a knowledge of the drive currents to the radiating elements and their relative positions in the array.

Definitions:

- $\mathbf{Z}$ = the impedance matrix of the array (n x n)
- $\mathbf{Z_B}$ = the output impedance matrix of the amplifiers (n x n with output impedances on the main diagonal; all other elements are zero)
- $\mathbf{i}$ = the current vector for the radiating elements (n x 1)
- e_r = $eA_r g_r Z_{Ar}$, the output e.m.f. supplied by the rth amplifier. (See Section 4).
- $\mathbf{E}$ = the vector of the e.m.f.s e_r (n x 1)
- $\mathbf{E_r}$ = the vector derived from $\mathbf{E}$ by putting all elements zero except for e_r.

Then
$$\mathbf{i} = (\mathbf{Z}+\mathbf{Z_B})^{-1}\mathbf{E}$$
$$= \sum(\mathbf{Z}+\mathbf{Z_B})^{-1}\mathbf{E_r}$$
$$= \sum e_r \mathbf{Y_r} \qquad (3)$$

where $\mathbf{Y_r}$ is the rth column of $(\mathbf{Z}+\mathbf{Z_B})^{-1}$.

If the current flowing in each radiating element is known (the current i_r flowing in the rth element), the current I in the receiving aerial is calculated from an expression of the form
$$I = \sum \mathrm{C_r} i_r$$

where the $\mathrm{C_r}$ depend upon the relative positions of the radiating elements in the array and the angular directions and distance of the receiving aerial from the array.[4] The above expression for I may also be written
$$I = \mathbf{C}\,\mathbf{i}$$

where $\mathbf{C}$ is a row vector (1 x n) containing the $\mathrm{C_r}$ in order.* Thus, using (3)
$$I = \mathbf{C}\sum e_r \mathbf{Y_r}$$
$$= \sum e_r \mathbf{C}\mathbf{Y_r}$$
$$= e\sum A_r g_r Z_{Ar}\mathbf{C}\mathbf{Y_r} \qquad (4)$$

Comparing (1) and (4) it will be seen that, in this example
$$B_r = \mathbf{C}\mathbf{Y_r}$$

The preceeding mathematical argument may be explained in the following manner. Referring to Fig. 4, suppose that just one of the output e.m.f.s produced by the amplifiers is applied, all the other e.m.f.s being zero. This single e.m.f. will generate currents in all the elements by mutual coupling and a radiation pattern will result. Similar patterns will exist for the remaining (n − 1) e.m.f.s applied singly; the complete radiation pattern being the sum of these patterns. To apply the reciprocity principle and obtain the equivalent receiving aerial directivity it is necessary to multiply each pattern by a ratio, which is Z_{Br}/Z_{Ar} for the rth pattern, before the summation is carried out.

* These two expressions for I are identical.

7. Discussion

7.1. It has been shown that a form of reciprocity principle may be applied to the study of active aerials. The principle may be employed to calculate the directivity pattern of an active receiving array from a consideration of a corresponding transmitting array. The simple reciprocity principle which is available for non-active systems does not hold in general in this case. Nevertheless, the more complex principle appropriate to active systems can be of value because transmitting cases are often the easiest to deal with theoretically.

7.2. It has appeared that the ratios of amplifier output and input impedances are crucial. It should not be thought, however, that these ratios provide useful design parameters because the A_r and B_r factors are functions of the amplifier impedances and must be recomputed for each new set of these.

7.3. One might wish to determine the effect of small changes in amplifier characteristics on the directivity pattern of an active receiving array even though the amplifiers were nominally identical. In this case the theory might be applied with the A_r and B_r factors expressed in perturbation form** so that variation limits on the pattern could be found.

8. References

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2. SHEA, T.E. 1929. Transmission networks and wave filters. New York, Van Nostrand, 1929.

3. SCHELKUNOFF, S.A. 1943. Electromagnetic waves. New York, Van Nostrand, 1943.

4. SCHELKUNOFF, S.A. 1943. A mathematical theory of linear arrays. *Bell Syst. Tech. J.*, 1943, **XXII**, 1, pp. 80 – 107.

** By perturbation form is meant an expression of the type $B_r' = B_r + \alpha_r \Delta Z_1 + \beta_r \Delta Z_2 + \gamma_r \Delta Z_3$ + etc. where ΔZ_1 etc. are small departures from the nominal valuea of Z_1 etc.

APPENDIX

Amplifiers with Output to Input Coupling

The theory requires that amplifiers in an active array can be represented by a small-signal equivalent circuit involving impedances and sources of e.m.f.

The proof of the reciprocity principle in Section 4 assumes an equivalent circuit as shown in Fig. 5(*a*).

An equivalent result can be derived if there is passive coupling between input and output as shown in Fig. 5(*b*).

The amplifier reversal is achieved by transferring the e.m.f. from the Z_B limb to the Z_A limb. In this case Equations (1) and (2) would have the form $I = I_o + \Sigma$ where I_o is the output current in the whole system in the absence of amplifier e.m.f.s. Z_A and Z_B may no longer be considered as the input and output impedances of the amplifier depending upon what definition is used for these quantities. Clearly more complex amplifier circuits could be introduced and results obtained on the lines laid down; but the increase in computational complexity would increase very rapidly.

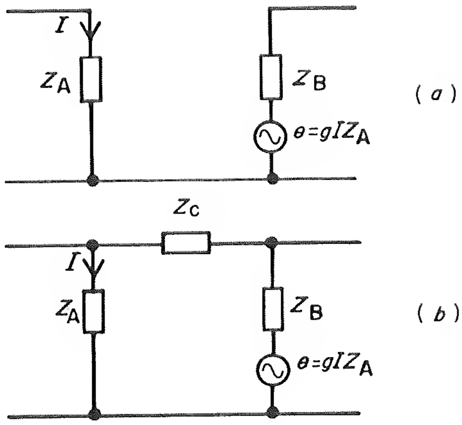

Fig. 5 - Equivalent amplifier circuits
(*a*) Equivalent circuit of buffer amplifier
(*b*) Equivalent circuit of amplifier with output to input coupling

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